NOTE

A CHARACTERIZATION OF COMPETITION GRAPHS

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Characterizations of competition graphs for arbitrary and acyclic directed graphs are presented.

Let $D$ be a directed graph having no multiple edges. The competition graph of $D$ is an undirected graph $G$ on the same node set as $D$ and having an undirected edge $\{v_i, v_j\}$ if and only if there exists a third node $v_k$ such that $(v_i, v_k)$ and $(v_j, v_k)$ are directed edges in the edge set of $D$. Competition graphs of acyclic digraphs were employed by Cohen \cite{1,2,3} to study ‘food web’ models in ecology. The various animal species in a geographic area were represented by the node set with a directed edge from node $v_i$ to node $v_j$ if species $i$ ‘preyed’ upon species $j$. Cohen observed that most food web models tend to be acyclic and justified the restriction to this class of digraphs. The competition graph of such a food web model then exhibits, by undirected edges, those species which compete for food. These graphs have also been studied by Roberts \cite{9,10} and more recently by Opsut \cite{8} who demonstrated that the problem of determining whether or not an arbitrary graph is the competition graph of some acyclic digraph is NP-complete.

Roberts has shown that any undirected graph $G$ can be a competition graph of an acyclic digraph if a sufficient number of isolated nodes are appended to $G$. Thus we may reject approaches which attempt to characterize competition graphs by ‘forbidden’ subgraphs. Our first result characterizes competition graphs of arbitrary digraphs (cycles and loops allowed). In the following $V(G)$ is the node set of $G$, $E(G)$ is the edge set and $\theta_1(G)$ is the minimal number of complete subgraphs which cover the edges.

**Theorem 1.** $G$ is the competition graph of an arbitrary digraph $D$ if and only if $\theta_1(G) \leq n$. 
Proof. With \( G \) the competition graph of \( D \) define, for \( 1 \leq i \leq n \), \( C_i \) as the subgraph of \( G \) induced by \( \{ v_j \mid (v_i, v_j) \in E(D) \} \). Clearly each \( C_i \) is a complete subgraph of \( G \) and every edge of \( G \) is in some \( C_i \). Thus \( \theta_1(G) = n \). Now assume \( \theta_1(G) = k \leq n \) and let \( C_1, C_2, \ldots, C_k \) be an edge cover of \( G \) by complete subgraphs. Construct \( D \) with \( V(D) = \{ v_1, v_2, \ldots, v_n \} \) and \( (v_i, v_j) \in E(D) \) if and only if \( v_i \in V(C_j) \). \( G \) is then the competition graph for \( D \).

Notice that \( D \), constructed as in the proof, may contain loops. For a characterization which does not allow loops see Roberts and Steif [11]. Other characterizations are given by Lundgren and Maybee [6].

We now consider the special case of characterizing competition graphs of acyclic digraphs. We shall need the following result [5].

**Lemma.** \( D \) is an acyclic digraph if and only if its nodes can be labeled so that
\[
(v_i, v_j) \in E(D) \quad \text{implies} \quad i < j.
\]

**Theorem 2.** The following statements are equivalent for an undirected graph \( G \) on \( n \) nodes:

(a) \( G \) is the competition graph of some acyclic directed graph \( D \).

(b) \( G \) has a vertex labeling \( u_1, u_2, \ldots, u_n \) so that there are complete subgraphs \( C_1, C_2, \ldots, C_n \) such that

(i) \( u_i \in V(C_j) \) implies \( i < j \), and

(ii) the \( C_i \)'s form an edge cover of \( G \).

(c) \( G \) has complete subgraphs \( C'_1, C'_2, \ldots, C'_{n-2} \) which form an edge cover of \( G \) such that \( |C'_1 \cup C'_2 \cup \cdots \cup C'_{j-1}| \leq j + 1 \) for \( 1 \leq i \leq n-2 \).

**Proof.** (a) \( \Rightarrow \) (b). Choose a vertex labeling \( u_1, u_2, \ldots, u_n \) of \( D \) as prescribed by the lemma. As before, for \( 1 \leq j \leq n \), define \( C_j \) as the subgraph of \( G \) induced by \( \{ v_i \mid (v_i, v_j) \in E(D) \} \). Clearly \( C_j \) is complete and (i) and (ii) are satisfied.

(b) \( \Rightarrow \) (c). The first condition of (b) implies \( C_1 \) and \( C_2 \) have no edges. Thus we may define, for \( 1 \leq i \leq n-2 \), \( C'_i = C_{i+2} \). For \( 1 \leq j \leq n-2 \), if \( v_i \in C'_1 \cup C'_2 \cup \cdots \cup C'_j \), then \( i \leq j + 1 \) by (i).

(c) \( \Rightarrow \) (a). Identify the nodes of \( G \) in the following way. Label as \( v_n \) a node not in \( \bigcup_{i=1}^{n-2} C'_i \), as \( v_{n-1} \) a different node not in \( \bigcup_{i=1}^{n-3} C'_i \), etc. Finally, arbitrarily label the remaining two nodes as \( v_2 \) and \( v_1 \). Let \( D \) be the directed graph on this set of nodes with \( E(D) = \{ (v_i, v_j) \mid v_i \in C'_{j-2} \} \).

\( G \) is easily seen to be the competition graph of \( D \). Furthermore, \( (v_i, v_j) \in E(D) \) implies \( i \leq j - 1 \). Thus \( D \) is acyclic by the lemma.

It is interesting to note that characterization (b) in Theorem 2 is similar to that
of Mukhopadhyay [7] for squares of graphs and Acharya and Vartak (as reported by Escalante, Montejano and Rojano [4]) for neighborhood graphs.

References